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schneider from the Parma codex, and it covers Ibn Ezra's introduction. The text of el-Bîrûnî is in a manuscript in the Bodleian Library, and has never been published.

THE THEORY OF SIMILAR FIGURES.

By ROGER A. JOHNSON, Western Reserve University.

The object of this paper is to place in a new and perhaps a more satisfactory light the elements of the well-known theory of two and of three directly similar figures in a plane. The attempt has been made to keep in the foreground the idea of determining each of such a set of figures by a base-line, that is, by one of a set of homologous line-segments. This notion clarifies the conception, and appreciably simplifies the treatment; further, as shown below, it enables us to avoid certain false proofs which are to be found in the usual discussions of this subject. The use of directed angles[1] is again advantageous, but of course not essential.

The fundamental operations of elementary plane geometry are four in number:

(*a*) *Translation,* or motion of a figure such that·every point is moved the same distance in the same direction,

(*b*) *Rotation* about a fixed point,

(*c*) *Reflection* with regard to a fixed line, which is the same in effect as turning the plane over on this line as axis,

(*d*) *Expansion* with regard to a fixed point, whereby the distance of each point of the figure from the fixed point is increased or decreased in a constant ratio (the same word *expansion* is used in all cases, whether the figure is actually enlarged or diminished).

If any figure is subjected to any succession of these operations, the resulting figure is similar to the original one; conversely it is easy to see that if two similar figures lie in a plane, one can be brought to coincide with the other by a combination, for instance, of a translation, a rotation, and an expansion. Our first object shall be to simplify this combination.

As a first lemma, we may note that if we operate on one of two directly similar figures in such a way that two of its points come to coincide with the corresponding points of the other, the figures coincide throughout.

Temporarily we shall use the word *homology* to designate the combination of a rotation and an expansion with regard to the same center.

THEOREM 1. *Given two line segments MN, $M'N'$, there exists a single operation, either a translation or a homology, that carries MN into $M'N'$.*

Case i. If MN and $M'N'$ are equal and parallel, and extend in the same direction, a translation that carries M into M' will carry N into N'.

Case ii. If $MNN'M'$ is a trapezoid, and MM' meets NN' at P, an expansion with regard to P, followed by a rotation through 180° when P is between the parallels MN and $M'N'$, carries M and N into M' and N' respectively.

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[1] "Directed Angles in Elementary Geometry," R. A. Johnson, this MONTHLY, March, 1917, p. 101 ff.

Case iii. If MN and $M'N'$ are segments of the same straight line not coming under case *i*, we may take a segment PQ, so that $M'N'QP$ is a rectangle. Let R be the point located as in case *ii* for MN and PQ, and let C be the foot of the perpendicular from R to the line $MNM'N'$. Then MN is carried into $M'N'$ by an expansion with regard to C, followed if necessary by a rotation of 180° about C.

Case iv. In the general situation, MN and $M'N'$ intersect at a point A. Let circles $MM'A$ and $NN'A$ intersect a second time at C; then triangles MNC and $M'N'C$ are directly similar, and if we rotate $M'N'C$ about C until CM' falls on CM, then expand until M' falls on M, we shall have moved N' to the position N.

In particular, one or two of the given points may fall at A: for instance, let A and M coincide; then for the circle $MM'A$ of the proof just given, we use the circle through M and M' which touches MN at M, and the proof holds as before. If both M and M' are at A, or if the circles are tangent there, then the point C is also there.

THEOREM 2. *If two directly similar figures lie in a plane,*

(*a*) *If they are congruent, with corresponding sides parallel and extending in the same direction, a translation will carry one into coincidence with the other.*

(*b*) *In every other case, there exist a point C, called the center of similitude, a number, the ratio of similitude, and an angle, the angle of similitude; if one figure is rotated about C through this angle, and expanded from C in this ratio, it is thereby made to coincide with the other figure.*

COROLLARY. The center of similitude is self-homologous in the two figures, and is the only self-homologous point; in other words, if we can determine a self-homologous point, we thereby find the center of similitude.

We notice a few interesting applications.

THEOREM 3. *If one vertex of a triangle of constant form is held fast, while a second vertex traces any figure whatever, the third vertex traces a similar figure.*

THEOREM 4. *If one of two directly similar figures is held motionless, while the other rotates about a fixed point, the center of similitude traces a straight line or a circle, according as the given figures are equal or unequal.*

COROLLARY. Obviously two circles may be regarded as similar figures in an infinite number of ways, with any point of the one homologous to any point of the other; the locus of the centers of similitude is a circle, the locus of a point whose distances from the centers of the given circles are proportional to the radii of the latter. This is called the *circle of similitude* of the two circles.

THEOREM 5. *If a figure is moved in a plane without change of form or size, so that two of its points remain constantly on fixed intersecting lines of the plane, the locus of the " instantaneous center of rotation " is a circle whose center is the intersection of the fixed lines.*

We may interpret these results in terms of maps of a plane region superposed, the same side up, on a table. We see that there is, except in one special case, one point whose images in the two maps coincide; this may be called the *double point* of the two maps. In practise this double point would of course often fall outside the boundaries of actual maps. If one map is held fast, while the other is rotated about a pivot, the double point traces a circle or a straight line, according as the maps are on different scales or the same scale. Our other theorems can as well be interpreted in this way.

We consider next the properties of three maps lying on a table; in other words, the relations among three directly similar figures in a plane. Each two of them determine a double point, or center of similitude. We will designate the figures by I, II, III; then the center of similitude of II and III shall be called C_1, that of III and I, C_2, and that of I and II, C_3. Thus C_1 is self-homologous in II and III; and so on.

It is immediately evident that the situation admits numerous special cases, for even in the simple case of two figures (Theorem 1) there are several cases to consider. We shall confine ourselves to the most general case; similar methods may be applied to any special case, yielding similar results. For instance, we shall exclude case *i* of Theorem 1; thus we shall always have three actual centers of similitude.

Let us then consider three similar figures, determined by homologous segments or base-lines, M_1N_1, M_2N_2, M_3N_3; let these lines be not concurrent nor parallel, but form a triangle $L_1L_2L_3$. If C_1, C_2, C_3 are the centers of similitude, triangles $M_2N_2C_1$ and $M_3N_3C_1$ are similar; and so on. We know that C_1 is the intersection of circles $L_1M_2M_3$ and $L_1N_2N_3$; and so on. Now the circles $L_1M_2M_3$, $L_2M_3M_1$, $L_3M_1M_2$ intersect at a point M, and the circles $L_1N_2N_3$, $L_2N_3N_1$, $L_3N_1N_2$ at a point N.[1]

THEOREM 6. *The points M and N lie on the circle or line $C_1C_2C_3$.*

For

$$\begin{aligned}\measuredangle\, C_2MC_3 &= \measuredangle\, C_2M,\ L_2C_2 + \measuredangle\, L_2C_2,\ L_3C_3 + \measuredangle\, L_3C_3,\ MC_3\\ &= \measuredangle\, MC_2L_2 + \measuredangle\, L_3C_3M + \measuredangle\, L_2C_2,\ L_3C_3\\ &= \measuredangle\, L_2C_2,\ L_3C_3.\end{aligned}$$

Similarly

$$\measuredangle\, C_2NC_3 = \measuredangle\, L_2C_2,\ L_3C_3.$$

If now L_2C_2 and L_3C_3 are not parallel, but meet at a point X, then the circle through C_3, M and N passes also through C_2, and therefore through C_1. X is also on this circle.

On the other hand, if L_2C_2 and L_3C_3 are parallel, the above equations show that M and N lie on the line C_2C_3, and it comes at once that C_1 is on the same line, and L_1C_1 is parallel to L_2C_2 and L_3C_3.

COROLLARY. We may construct a system wherein the centers of similitude are collinear, or a system wherein they lie on an actual circle.

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[1] *Loc. cit.*, p. 103.

For if the triangle $L_1L_2L_3$ and segments M_2N_2 and M_3N_3 are given, C_1 can be found. We may put C_2 anywhere, then C_3, M_1 and N_1 can be located. According as we make C_1L_1 and C_2L_2 parallel or intersecting, $C_1C_2C_3$ will be a line or a finite circle.

In the sequel we shall confine our attention to the case that C_1, C_2, C_3 lie on a circle of finite radius; the reader can easily deduce the corresponding theorems when they are collinear. The circle $C_1C_2C_3$ is called the *circle of similitude* of the three figures, and the triangle $C_1C_2C_3$ is called the *triangle of similitude.*

THEOREM 7. *The lines L_1C_1, L_2C_2, L_3C_3 are concurrent at a point X on the circle of similitude. That is, the triangle of any three homologous lines is in perspective with the triangle of similitude; the center of perspective lies on the circle of similitude.*[1]

For we have just seen that L_2C_2 and L_3C_3 meet at a point X on the circle of similitude; each of these must also meet L_1C_1 on the same circle. But L_3C_3, for instance, meets the circle at C_3 and at only one other point; in general, L_1C_1 and L_2C_2 do not pass through C_3; hence all three of these lines are concurrent at X.

THEOREM 8. *If the lines XL_1, XL_2, XL_3 are divided proportionally at L_1', L_2', L_3'; then $L_2'L_3'$, $L_3'L_1'$, $L_1'L_2'$ are parallel respectively to L_2L_3, L_3L_1, L_1L_2, and are homologous lines in the similar figures. In particular, lines through X and parallel to L_2L_3, L_3L_1, L_1L_2 respectively constitute a set of concurrent homologous lines.*

THEOREM 9. *Conversely, if three homologous lines are concurrent, the point of concurrence lies on the circle of similitude.*

The easiest proof is made by returning to first principles. Let M_1N_1, M_2N_2, M_3N_3 be homologous segments whose lines are concurrent at Z_1. Let an inversion with regard to Z as center carry M_1, N_1, etc., into M_1', N_1', etc. We now have two triangles $M_1'M_2'M_3'$, $N_1'N_2'N_3'$ in perspective at Z; therefore the intersections of corresponding sides are collinear. Let $M_2'M_3'$ meet $N_2'N_3'$ at C_1'; then inverting back to the original figure, the inverse of this point C_1' is the intersection of circles M_2M_3Z and N_2N_3Z, or in other words the center of similitude C_1. But if C_1', C_2', C_3' lie on a line, then C_1, C_2, C_3 lie on a circle through the center of inversion Z, and the proof is complete.

THEOREM 10. *There are three fixed points on the circle of similitude; any set of concurrent homologous lines pass respectively through these points. They are called the invariable points of the system.*

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[1] So far as the writer can ascertain, this well-known theorem has never been proved. The proofs, as given by McClelland and Lachlan, for instance, are based on a false theorem namely: "if two triangles are $A_1A_2A_3$ and PQR; if the lengths of the perpendiculars from P to A_2A_3, A_3A_1, A_1A_2 are p_1, p_2, p_3, and so on; then if A_1P, A_2Q, A_3R are concurrent,

$$\frac{p_2q_3r_1}{p_3r_1q_2} = 1$$

and conversely." The converse, which is used in our theorem, is false: for example, let P and Q lie on the bisectors of the interior angles at A_1 and A_2 respectively, while R is on the exterior bisector of A_3; the equation is satisfied by the perpendiculars, but the triangles are not in perspective. The same flaw vitiates the current proofs of several other standard theorems of this type.

For if XP_2 and XP_3 are homologous lines of II and III, meeting at X on the circle of similitude, and cutting that circle again at T_2 and T_3 respectively, then each of the angles $\measuredangle C_1XP_2$, $\measuredangle C_1XP_3$ is constant and therefore arcs C_1T_2 and C_1T_3 are constant, and T_2, T_3 are fixed points.

THEOREM 11. *The invariable points are homologous; they form a triangle inversely similar to that formed by three homologous lines.*

For (*a*) triangles $C_1T_2T_3$ and $C_2P_2P_3$ are similar, (*b*) $\measuredangle T_1T_3T_2 = \measuredangle T_1XT_2$, etc. Let C_2T_2 meet C_3T_3 at M.

THEOREM 12. *The point C_1', homologous in I to the double point C_1 of II and III, lies on the circle C_2C_3M.*

For

$$\measuredangle C_2C_1'T_1 = \measuredangle C_2C_1T_3 = \measuredangle C_2T_2T_3$$

$$\measuredangle T_1C_1'C_3 = \measuredangle T_2C_1C_3 = \measuredangle T_2T_3C_3$$

whence

$$\measuredangle C_2C_1'C_3 = \measuredangle C_2T_2,\ C_3T_3.$$

THEOREM 13. *Both C_1' and M lie on C_1T_1.*

For

$$\measuredangle C_3T_1C_1' = \measuredangle C_3T_3C_1 = \measuredangle C_3T_1C_1$$

and

$$\measuredangle C_3C_1'M = \measuredangle C_3C_2M = \measuredangle C_3C_2T_2 = \measuredangle C_3T_1T_2 = \measuredangle C_3C_1'T_1.$$

Our results may be summed up as follows:

The invariable triangle and the triangle of similitude are inscribed in the same circle and are in perspective at a point M. The lines C_1T_1, C_2T_2, C_3T_3, which are concurrent at M, meet again the respective circles C_2C_3M, C_3C_1M, C_1C_2M at C_1', C_2', C_3' respectively.

Concerning the properties of triads of homologous points, we merely state the main theorems; proofs by means of directed angles are not difficult.

THEOREM 14. *If P_1, P_2, P_3 are homologous, circles $C_2C_3P_1$, $C_3C_1P_2$, $C_1C_2P_3$ are concurrent at a point X, for which*

$$\measuredangle C_2XC_3 = \measuredangle C_2MC_3 + \measuredangle P_3P_1P_2.$$

COROLLARY. If homologous points P_1, P_2, P_3 move so that $\measuredangle P_3P_1P_2$ remains equal to a given angle α, then each moves on a circle; in particular, P_1 lies on that circle through C_2 and C_3 for which

$$\measuredangle C_2P_1C_3 = \measuredangle C_2MC_3 + \alpha.$$

THEOREM 15. *The loci of collinear homologous points are respectively the circles C_2C_3M, C_3C_1M, C_1C_2M; and the line through the three points passes also through M.*

THEOREM 16. *There is a single triad of homologous points which are the vertices of a triangle similar in a given sense to a given triangle.*

The construction is effected as follows: let $V_1V_2V_3$ be the given triangle. Let X be so located that

$$\sphericalangle C_2XC_3 = \sphericalangle C_2MC_3 + \sphericalangle V_3V_1V_2$$

with similar expressions for $\sphericalangle C_3XC_1$ and $\sphericalangle C_1XC_2$. There is just one such point X. Now regard circle C_2XC_3 as a part of Fig. 1, and find the homologous circle of II, then P_2, one of the required triad of points, lies on this circle. But P_2 lies also on circle C_3XC_1, and therefore it is uniquely determined.

THEOREM 17. *Similarly, there is a single triad of collinear homologous points, whose distances are in the same ratios as the distances among three given collinear points.*[1]

It is well known that in the special case when three similar figures are constructed on the sides of a triangle $A_1A_2A_3$, in such a way that A_2A_3, A_3A_1, A_1A_2 are homologous segments, then

(*a*) the triangle of similitude is the second Brocard triangle,
(*b*) the invariable triangle is the first Brocard triangle,
(*c*) the circle of similitude is therefore the Brocard circle,
(*d*) the point M is the median point,
(*e*) the circles MC_2C_3, etc., are the Neuberg circles.

NOTE ON A CERTAIN CLASS OF DETERMINANTS.

By W. H. METZLER, Syracuse University.

In this journal for May, 1915, there appeared the translation of a paper by Professor Pascal which considered a class of determinants which have the property of being expressible as the sum of two squares, and it is the main object of this note to point out another proof of this fact.

The determinants in question are of the form:

$$D \equiv \begin{vmatrix} a_{11} & \cdots & a_{1n} & -b_{11} & \cdots & -b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} & -b_{n1} & \cdots & -b_{nn} \\ b_{11} & \cdots & b_{1n} & a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{nn} & a_{n1} & \cdots & a_{nn} \end{vmatrix},$$

which is the determinant of the set of linear homogeneous complex equations for the $2n$ quantities $p_1, \cdots, p_n, q_1, \cdots, q_n$:

$$\begin{array}{c} (a_{11} + b_{11}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{1n} + b_{1n}\cdot i)(p_n + q_n\cdot i) = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (a_{n1} + b_{n1}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{nn} + b_{nn}\cdot i)(p_n + q_n\cdot i) = 0. \end{array} \tag{1}$$

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[1] Theorems 16 and 17 are doubtless well known, but the author has not chanced to see them explicitly stated. But see McClelland.